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PERIODIC PROBLEM FOR AN ELASTIC PLANE WITH THIN-WALLED INCLUSIONS

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The state of stress of a homogeneous and piecewise-homogeneous plane with a thin-walled elastic inclusion of finite length was investigated in [1 – 4]. The main attention in [1, 2] was paid to determining the difference in the shear stresses on opposite edges of the inclusion under the assumption that the displacements are equal. Consequently, the solution of the Prandtl integral equation, to which the problem reduced, is suitable for flexible inclusions with a sufficiently high elastic modulus. The approach in [3, 4] for the case of a homogeneous plane was somewhat different, where the relative displacement of the inclusion edges was taken into account while the difference in the originating stresses was neglected. Some results for a corresponding periodic problem were obtained in [5, 6].

The periodic problem of elastic equilibrium of a plane consisting of two bonded isotropic half-planes with finite thin-walled elastic inclusions on the straight line of separation of the materials is studied. A system of singular integral equations is obtained, whose solution is suitable for inclusions of any stiffness from absolutely stiff or flexible but inextensible to absolutely yielding (slits). Therefore, a close connection is established between crack theory and the theory of thin-walled elastic inclusions. In the limit case, the solution is obtained for the problem of one inclusion in a more accurate formulation than in [1 – 4].

1. The elastic equilibrium of a plane consisting of two bonded isotropic half-planes with thin-walled elastic inclusions of length $2a$ and width $2h$, arranged on the straight line separating the materials with the period 2π (Fig. 1), is considered. It is assumed that the half-planes are subjected to stresses at infinity ($\sigma_y^\infty = P$, $\sigma_{x1}^\infty = P_1$, $\sigma_{x2}^\infty = P_2$) as well as to the system of p^1 groups of concentrated forces $Q_j = Q_{xj} + iQ_{yj}$ acting at the points $z_{0j} + 2n\pi$ ($j = 1, 2, \ldots, p^1$; $n = 0, \pm1, \pm2, \ldots$) and p^2 groups of moments M_k applied at the points $z_{k0} + 2n\pi$ ($k = 1, 2, \ldots, p^2$; $n = 0, \pm 1, \pm2, \ldots$). The first m^1 groups of forces and m^2 groups of moments are applied at points of the lower half-plane, and the rest in the upper half-plane. Such a load selection permits considering the problem periodic since the stress-strain state is identical at the points $z + 2n\pi$ ($n = 0, \pm 1, \pm2, \ldots$). Therefore, we can henceforth limit ourselves to analyzing the strip $|x| \leqslant \pi$, $|y| < \infty$.

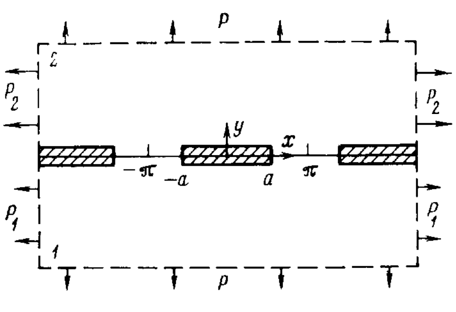

Fig. 1

Let L_1 denote the set of intervals $[2n\pi - a,\ 2n\pi + a]$ $(n = 0, \pm 1, \pm 2, \ldots)$ occupied by the inclusions, and L_2 the rest of the real axis.

Taking account of the small thickness of the inclusions, let us write the boundary conditions on the line $y = 0$ as (1.1)

$$(\sigma_{y1} - i\,\tau_{xy1}) - (\sigma_{y2} - i\,\tau_{xy2}) = f_1(x) - i f_2(x) = \begin{cases} f_1^*(x) - i f_2^*(x), & x \in L_1 \\ 0, & x \in L_2 \end{cases}$$

$$(u_1' + i v_1') - (u_2' + i v_2') = f_3(x) + i f_4(x) = \begin{cases} f_3^*(x) + i f_4^*(x), & x \in L_1 \\ 0, & x \in L_2 \end{cases}$$

$$f_j(x) = f_j(x + 2n\pi), \quad j = 1, 2, 3, 4; \quad n = 0, \pm 1, \pm 2, \ldots$$

Under the assumption that the transverse strains are equal at the edges of the inclusions we have, by analogy with [7], four conditions for interaction between the thin-walled inclusions and the surrounding medium

$$u_1' + u_2' = 2k_0\sigma_x - k_1(\sigma_{y1} + \sigma_{y2}) \tag{1.2}$$

$$u_2 - u_1 = \frac{h}{\mu_0}(\tau_{xy1} + \tau_{xy2}) - h(v_1' + v_2')$$

$$u_2' - u_1' = -k_2(\sigma_{y2} - \sigma_{y1})$$

$$v_2 - v_1 = k_0 h(\sigma_{y1} + \sigma_{y2}) - 2k_1 h\sigma_x, \quad x \in L_1$$

Here

$$\sigma_x = N_- + \frac{1}{2h}\int_{2n\pi - a}^{x} [\tau_{xy2}(t) - \tau_{xy1}(t)]\, dt, \quad 2n\pi - a \leqslant x \leqslant 2n\pi + a \tag{1.3}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$k_0 = \frac{1}{E_0}, \quad k_1 = \frac{\nu_0}{E_0}, \quad k_2 = \frac{k_1^2 - k_0^2}{k_1}, \quad \mu_0 = \frac{E_0}{2(1 + \nu_0)} \tag{1.4}$$

The subscript 1 refers to the lower half-plane, the subscript 2 to the upper half-plane, τ_{xyi}, σ_{yi}, σ_{xi} are the shear and normal stresses, u_i, v_i are components of the displacement vector, E_0, ν_0 are the elastic modulus and Poisson's ratio of the material of the inclusions, N_- is the normal stress on the left endfaces of the inclusions, and $f_j(x)$ are periodic functions to be determined.

2. The stresses and displacements are given by the formula [8]

$$\sigma_{yj} - i\,\tau_{xyj} = \Phi_j(z) + \overline{\Phi_j(z)} + z\,\overline{\Phi_j'(z)} + \overline{\Psi_j(z)} \tag{2.1}$$

$$2\mu_j(u_j' + i v_j') = \varkappa_j\Phi_j(z) - \overline{\Phi_j(z)} - z\,\overline{\Phi_j'(z)} - \overline{\Psi_j(z)}, \quad j = 1, 2$$

$$\Phi_j(z) = \Sigma_j^1 S_{jk}(z) + \Gamma_j + \Phi_{0j}(z)$$

$$\Psi_j(z) = \Sigma_j^1 R_{jk}(z) + \Sigma_j^2 R_k^*(z) + \Gamma_j' + \Psi_{0j}(z)$$

$$S_{jk}(z) = -e_j \sum_{n=-\infty}^{\infty} \frac{Q_k}{z + 2n\pi - z_{0k}}$$

$$R_{jk}(z) = e_j \sum_{n=-\infty}^{\infty} \left[\frac{\varkappa_j \overline{Q}_k}{z + 2n\pi - z_{0k}} - \frac{(\bar{z}_{0k} + 2n\pi)\, Q_k}{(z + 2n\pi - z_{0k})^2}\right]$$

$$R_k^*(z) = \frac{iM_k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{z + 2n\pi - z_{k0}}, \quad e_j = \frac{1}{2\pi(1 + \varkappa_j)}$$

$$\varkappa_j = \frac{3-\nu_j}{1+\nu_j}, \quad \mu_j = \frac{E_j}{2(1+\nu_j)}$$

$$\Sigma_1{}^i = \sum_{k=1}^{m^i}, \quad \Sigma_2{}^i = \sum_{k=m^i+1}^{p^i}, \quad \Gamma_j = \frac{1}{4}(P+P_j),$$

$$\Gamma_j' = \frac{1}{2}(P-P_j), \quad i, j = 1, 2$$

$\Phi_{0j}(z)$, $\Psi_{0j}(z)$ are functions holomorphic outside L_1 which vanish as $z \to x + i\infty$.

Let us determine $\Phi_1(z)$ in the upper half-plane S_2 and $\Phi_2(z)$ in the lower half-plane S_1 by the relationship

$$\Phi_j(z) = -\overline{\Phi}_j(z) - z\overline{\Phi}_j'(z) - \overline{\Psi}_j(z), \qquad z \in S_k;\ j,\ k = 1,2;\ k \neq j$$

and by using formula (1.421.3) in [9] we obtain on the real axis

$$\sigma_{y1} - i\tau_{xy1} = \Phi_{01}^-(x) - \Phi_{01}^+(x) + A_1(x) \tag{2.2}$$

$$2\mu_1(u_1' + iv_1') = \varkappa_1\Phi_{01}^-(x) + \Phi_{01}^+(x) + B_1(x)$$

$$\sigma_{y2} - i\tau_{xy2} = \Phi_{02}^+(x) - \Phi_{02}^-(x) + A_2(x), \quad 2\mu_2(u_2' + iv_2') = \varkappa_2\Phi_{02}^+(x) + \Phi_{02}^-(x) + B_2(x)$$

$$A_j(x) = A_j + e_j\Sigma_j{}^1[(\varkappa_jQ_k - \overline{Q}_k)\overline{L_k(x)} - Q_kL_k(x) + \overline{Q}_kC_k(x)] + \Sigma_j{}^2\frac{iM_k}{2\pi}L_k^*(x)$$

$$B_j(x) = B_j + e_j\Sigma_j{}^1[(\overline{Q}_k - \varkappa_jQ_k)\overline{L_k(x)} - \varkappa_jQ_kL_k(x) - \overline{Q}_kC_k(x)] - \Sigma_j{}^2\frac{iM_k}{2\pi}L_k^*(x)$$

$$A_j = \Gamma_j + \overline{\Gamma}_j + \overline{\Gamma}_j', \quad B_j = \varkappa_j\Gamma_j - \overline{\Gamma}_j - \overline{\Gamma}_j'$$

$$L_k(x) = \frac{1}{2}\operatorname{ctg}\left(\frac{x - z_{0k}}{2}\right), \quad L_k^*(x) = \frac{1}{2}\operatorname{ctg}\left(\frac{x - z_{k0}}{2}\right)$$

$$C_k(x) = \frac{1}{4}(\overline{z}_{0k} - z_{0k})\operatorname{cosec}^2\left(\frac{x - z_{0k}}{2}\right) + \overline{L_k(x)}$$

Using the relationship (2.2) and solving the appropriate conjugate problem we find from conditions (1.1) after some manipulation

$$\Phi_{01}^+(x) = \frac{\mu_1}{c_{21}}[2\mu_2X^+(x) + \varkappa_2Y^+(x)] \tag{2.3}$$

$$\Phi_{01}^-(x) = \frac{\mu_1}{c_{12}}[-2\mu_2X^-(x) + Y^-(x)]$$

$$\Phi_{02}^+(x) = \frac{\mu_2}{c_{21}}[-2\mu_1X^+(x) + Y^+(x)]$$

$$\Phi_{02}^-(x) = \frac{\mu_2}{c_{12}}[2\mu_1X^-(x) + \varkappa_1Y^-(x)]$$

$$X^\pm(x) = \pm\frac{1}{2}[2\omega(x) + f_3(x) + i f_4(x)] + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2\omega(t) + f_3(t) + i f_4(t)}{t - x}dt$$

$$Y^\pm(x) = \pm\frac{1}{2}[\Omega(x) - f_1(x) + i f_2(x)] + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Omega(t) - f_1(t) + i f_2(t)}{t - x}dt$$

$$\omega(x) = \frac{B_2(x)}{4\mu_2} - \frac{B_1(x)}{4\mu_1}, \quad \Omega(x) = A_1(x) - A_2(x), \quad c_{12} = \mu_1 + \varkappa_1\mu_2$$
$$c_{21} = \mu_2 + \varkappa_2\mu_1$$

Substituting (2.3) into (2.2), separating real and imaginary parts, and also taking account of the periodicity of the functions $f_j(x)$, we obtain expressions for the stresses and the derivatives of the displacements

$$\sigma_{y1}(x) = \sigma_y^\circ(x) + m_{12}^+ f_1(x) + l_1^- f_3(x) + m_{12}^- t_2(x) - l_1^+ t_4(x) \quad (2.4)$$
$$\sigma_{y2}(x) = \sigma_{y1}(x) - f_1(x)$$
$$\tau_{xy1}(x) = \tau_{xy}^\circ(x) + m_{12}^+ f_2(x) - l_1^- f_4(x) - m_{12}^- t_1(x) - l_1^+ t_3(x), \quad \tau_{xy2}(x) = \tau_{xy1}(x) - f_2(x)$$
$$\sigma_{x1}(x) = \sigma_{x1}^\circ(x) + n_{12}^- f_1(x) + r_{12}^+ f_3(x) + n_{12}^+ t_2(x) - r_{12}^- t_4(x), \quad |x| \leqslant \pi$$
$$\sigma_{x2}(x) = \sigma_{x2}^\circ(x) - n_{21}^- f_1(x) - r_{21}^+ f_3(x) + n_{21}^+ t_2(x) - r_{21}^- t_4(x)$$
$$u_1'(x) = u^{\circ\prime}(x) + l_2^- f_1(x) + m_{21}^+ f_3(x) + l_2^+ t_2(x) + m_{21}^- t_4(x), \quad u_2'(x) = u_1'(x) - f_3(x)$$
$$v_1'(x) = v^{\circ\prime}(x) - l_2^- f_2(x) + m_{21}^+ f_4(x) + l_2^+ t_1(x) - m_{21}^- t_3(x)$$
$$v_2'(x) = v_1'(x) - f_4(x)$$

Here

$$\sigma_y^\circ(x) = \operatorname{Im}[I_1(x)] + \operatorname{Re}[I_3(x)] \quad (2.5)$$
$$\tau_{xy}^\circ(x) = \operatorname{Re}[I_1(x)] - \operatorname{Im}[I_3(x)]$$
$$\sigma_{xj}^\circ(x) = \operatorname{Im}[I_{jn}(x)] + 2\operatorname{Re}[-2e_j\Sigma_j^1 Q_k L_k(x) + 2\Gamma_j - (-1)^j r_{jn}\,\omega(x) + q_{nj}A_j(x) + p_{jn}A_n(x)]$$
$$u^{\circ\prime}(x) = \operatorname{Im}[I_2(x)] + \operatorname{Re}[I_4(x)]$$
$$v^{\circ\prime}(x) = -\operatorname{Re}[I_2(x)] + \operatorname{Im}[I_4(x)]$$

$$t_i(x) = \frac{1}{2\pi}\int_{-a}^{a} f_j(t)\operatorname{ctg}\left(\frac{t-x}{2}\right)dt, \quad I_{jn}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n_{jn}^+\Omega(t) - 2r_{jn}^-\omega(t)}{t-x}dt$$

$$I_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m_{12}^-\Omega(t) - 2l_1^+\omega(t)}{t-x}dt, \quad I_2(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{l_2^+\Omega(t) + 2m_{12}^-\omega(t)}{t-x}dt$$

$$I_3(x) = m_{21}^+ A_1(x) + m_{12}^+ A_2(x) + 2l_1^-\omega(x)$$
$$I_4(x) = \frac{m_{12}^+}{2\mu_1}B_1(x) + \frac{m_{21}^+}{2\mu_2}B_2(x) - l_2^-\Omega(x)$$
$$q_{jn} = -\frac{\mu_n}{c_{nj}} - \frac{1}{2}m_{jn}^+, \quad p_{jn} = \frac{\mu_j}{c_{jn}} - \frac{1}{2}m_{jn}^+$$
$$r_{jn}^\pm = \mu_1\mu_2\frac{3c_{nj} \pm c_{jn}}{c_{12}c_{21}}, \quad m_{jn}^\pm = \mu_i\frac{c_{nj} \pm c_{jn}\varkappa_n}{2c_{12}c_{21}}$$
$$n_{jn}^\pm = \mu_j\frac{3c_{nj} \pm c_{jn}\varkappa_n}{2c_{12}c_{21}}, \quad l_1^\pm = \mu_1\mu_2\left(\frac{1}{c_{12}} \pm \frac{1}{c_{21}}\right)$$
$$l_2^\pm = \frac{c_{21}\varkappa_1 \pm c_{12}\varkappa_2}{4c_{12}c_{21}}, \quad i = 1, 2, 3, 4;\ j, n = 1, 2;\ n \neq j$$

(σ_y°, σ_{x1}°, σ_{x2}°, τ_{xy}°, $u^{\circ\prime}$, $v^{\circ\prime}$ are the appropriate stresses and derivatives of the displacements on the line joining the half-planes in the absence of inclusions).

Substitution of (2.4) into condition (1.2) results in a system of integral equations

$$t_2(x) + \lambda_{11}t_4(x) + \lambda_{12}f_1(x) + \lambda_{13}f_3(x) - \lambda_1 \int_{-a}^{x} f_2(t)\,dt = F_1(x) \quad (2.6)$$

$$t_3(x) + \lambda_{21}t_1(x) + \lambda_{22}f_2(x) + \lambda_{23}f_4(x) - \lambda_2 \int_{-a}^{x} f_3(t)\,dt = F_2(x)$$

$$t_4(x) + \lambda_{31}t_2(x) + \lambda_{32}f_3(x) + \lambda_{33}f_1(x) + \lambda_3 \int_{-a}^{x} f_2(t)\,dt + \lambda_4 \int_{-a}^{x} f_4(t)\,dt = F_3(x)$$

$$f_3(x) = -k_2 f_1(x), \quad |x| \leqslant a < \pi$$

Here

$$F_1(x) = \frac{1}{\Lambda_1}[k_0 N_- - u^{\circ\prime}(x) - k_1\sigma_y^\circ(x)] \quad (2.7)$$

$$F_2(x) = \frac{\mu_0}{\Lambda_2}\left[v^{\circ\prime}(x) - \frac{\overset{\circ}{\tau}_{xy}(x)}{\mu_0} - \frac{c_1^-}{2h}\right]$$

$$F_3(x) = \frac{1}{l_1^+}\left[\sigma_y^\circ(x) + \frac{c_2^-}{2hk_0} - \nu_0 N_-\right]$$

$$\lambda_{11} = \frac{m_{21}^- - k_1 l_1^+}{\Lambda_1}, \quad \lambda_{12} = \frac{l_2^- + k_1\Lambda_3}{\Lambda_1}, \quad \lambda_{13} = \frac{k_1 l_1^- + \Lambda_3}{\Lambda_1}$$

$$\lambda_{21} = -\frac{m_{12}^+ + \mu_0 l_2^+}{\Lambda_2}$$

$$\lambda_{22} = \frac{l_2^-\mu_0 - \Lambda_3}{\Lambda_2}, \quad \lambda_{23} = \frac{l_1^- + \mu_0\Lambda_3}{\Lambda_2}, \quad \lambda_{31} = -\frac{m_{12}^-}{l_1^+}$$

$$\lambda_{32} = -\frac{l_1^-}{l_1^+}, \quad \lambda_{33} = \frac{\Lambda_3}{l_1^+}$$

$$\lambda_1 = \frac{k_0}{2h\Lambda_1}, \quad \lambda_2 = -\frac{\mu_0}{2h\Lambda_2}, \quad \lambda_3 = \frac{\nu_0}{2hl_1^+}, \quad \lambda_4 = -\frac{1}{2hk_0 l_1^+}$$

$$\Lambda_1 = l_2^+ + k_1 m_{12}^-, \quad \Lambda_2 = m_{12}^-\mu_0 - l_1^+, \quad \Lambda_3 = m_{21}^+ - \frac{1}{2}$$

The stresses at infinity and the elastic constants of the materials are connected by the relationship (2.8) and the desired functions by conditions (2.9)

$$\mu_2 B_1 = \mu_1 B_2 \quad (2.8)$$

$$\int_{-a}^{a} f_i(t)\,dt = A^i, \quad i = 1, 2, 3, 4 \quad (2.9)$$

$$A^1 = 0, \quad A^2 = 2h(N_+ - N_-), \quad A^3 = c_1^+ - c_1^-, \quad A^4 = c_2^+ - c_2^-$$

The normal stresses at the endfaces of the inclusions $N_\pm$ are determined by the formula

$$N_\pm = \frac{1}{2}[\eta_1\sigma_{x1}^\circ(x) + \eta_2\sigma_{x2}^\circ(x)]\,|_{x=\pm a}, \quad \eta_i = \left[\frac{\min(\mu_0, \mu_i)}{\mu_i}\right]^{1/2}, \quad i = 1, 2 \quad (2.10)$$

The quantities $c_1^\pm$ and $c_2^\pm$ characterize the displacement of the lower edge of the inclusion endface $x = \pm a$ relative to the upper edge of the same endface in the ox and oy directions, respectively, and are evaluated by means of the formulas

$$c_1^{\pm} = -2h\left\{(1+\nu_0)\,\tau_{xy}^{\circ}(x)\left[\frac{1}{\max(E_0, E_1)} + \frac{1}{\max(E_0, E_2)}\right] - v^{\circ\prime}(x)\right\}\Bigg|_{x=\pm a} \tag{2.11}$$

$$c_2^{\pm} = -h\left[\frac{\sigma_y^{\circ}(x) - \nu_0\sigma_{x1}^{\circ}(x)}{\max(E_0, E_1)} + \frac{\sigma_y^{\circ}(x) - \nu_0\sigma_{x2}^{\circ}(x)}{\max(E_0, E_2)}\right]\Bigg|_{x=\pm a}$$

Because of the interface conditions, the normal and shear stresses (with the exception of $\sigma_{xi}(x)$), as well as the derivatives of the displacements of the edges of the inclusions are described by (2.4), where the subscripts 1 and 2 should be referred to the lower and upper edges of the inclusions, respectively. The mean normal stress in the transverse section of the inclusion σ_x is determined by (1.3).

In the case of absolutely stiff inclusions $(E_0 \to \infty)$, we have $f_3(x) = f_4(x) = 0$, and the system (2.6) goes over into one equation to determine the functions $f_1(x) + if_2(x)$

$$\frac{1}{2\pi i}\int_{-a}^{a}[f_1(t) + if_2(t)]\operatorname{ctg}\left(\frac{t-x}{2}\right)dt + \frac{l_2^-}{l_2^+}[f_1(x) + if_2(x)] = \tag{2.12}$$

$$-\frac{1}{l_2^+}[u^{\circ\prime}(x) - iv^{\circ\prime}(x)], \qquad |x| \leqslant a$$

When $E_0 = 0$, we obtain $f_1(x) = f_2(x) = 0$ and the integral equation of the periodic problem for the system of slits on the line separating the two plane media is

$$\frac{1}{2\pi i}\int_{-a}^{a}[f_3(t) + if_4(t)]\operatorname{ctg}\left(\frac{t-x}{2}\right)dt - \frac{l_1^-}{l_1^+}[f_3(x) + if_4(x)] = \tag{2.13}$$

$$\frac{1}{l_1^+}[\sigma_y^{\circ}(x) - i\,\tau_{xy}^{\circ}(x)], \quad |x| \leqslant a$$

If the inclusions have a negligibly small bending stiffness, then we can *a priori* put $\sigma_{y1} = \sigma_{y2}$. The fourth equation in the system (2.6) should be eliminated from consideration and it should be considered that $f_1(x) = 0$. In this case we have $f_4(x) = 0$ as $E_0 \to \infty$ and the equation corresponding to the periodic problem of inextensible filaments mounted on a straight line separating the two materials is

$$\frac{1}{2\pi i}\int_{-a}^{a}[f_2(t) + i\Lambda_4 f_3(t)]\operatorname{ctg}\left(\frac{t-x}{2}\right)dt + \frac{l_2^-\Lambda_4}{m_{12}^-}[f_2(x) + i\Lambda_4 f_3(x)] = \tag{2.14}$$

$$\frac{\Lambda_4}{m_{12}^-}v^{\circ\prime}(x) + \frac{i}{l_2^+}u^{\circ\prime}(x)$$

$$\Lambda_4 = \sqrt{-\lambda_3 m_{12}^- / l_2^- l_2^+}, \qquad |x| \leqslant a$$

The integral equations (2.12) – (2.14) have identical structure and are solved in closed form [10].

3. Let the half-planes have identical mechanical characteristics ($E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$). Then

$$\lambda_{12} = \lambda_{13} = \lambda_{22} = \lambda_{23} = \lambda_{32} = \lambda_{33} = 0 \tag{3.1}$$

in the system (2.6).

Using the results in [11], where an integro-differential equation with a Hilbert kernel was solved in particular, let us seek the solution of the system (2.6) in the case (3.1) as the series

$$f_i(x) = \sec\frac{x}{2}\,[2(\cos x - \cos a)]^{-1/2}\sum_{n=0}^{\infty} A_n{}^i T_n\left(\operatorname{tg}\frac{x}{2}\operatorname{ctg}\frac{a}{2}\right) \tag{3.2}$$

$$i = 1, 2, 3, 4; \quad |x| \leqslant a$$

whose coefficients are determined from the infinite systems of linear algebraic equations (3.3), (3.4) and the relationships (3.5)

$$\frac{\pi}{2}A^2_{2k+1+p} + \sum_{n=0}^{\infty}\left[r_1 B_{2n+p,\,2k+p} + r_2 \sum_{j=0}^{\infty}{}' B_{2n+p,\,2j+p}B_{2j+p,\,2k+p}\right]A^2_{2n+1+p} = \Phi_{2k+p} \tag{3.3}$$

$$\frac{\pi}{2}A^3_{2k+1+p} + \lambda\sum_{n=0}^{\infty}{}' B_{2n+p,2k+p}A^3_{2n+1+p} = \Phi^2_{2k+p}, \qquad p = 0, 1; \; k = 0, 1, \ldots \tag{3.4}$$

$$A^1_{2k+1+p} = -\frac{1}{k_2}A^3_{2k+1+p} \tag{3.5}$$

$$A^4_{2k+1+p} = \frac{2}{\pi\lambda_{11}}\left[\Phi^3_{2k+p} - \frac{\pi}{2}A^2_{2k+1+p} - \lambda_1'\sum_{n=0}^{\infty} B_{2n+p,\,2k+p}A^2_{2n+1+p}\right]$$

Here

$$\Phi_l{}^1 = 2\sin\frac{a}{2}g_l{}^1 + C_l{}^1 A_0{}^\circ + C_l{}^2 A_0{}^2, \qquad \Phi_l{}^2 = 2\sin\frac{a}{2}g_l{}^2 + C_l{}^3 A_0{}^1$$

$$\Phi_l{}^3 = 2\sin\frac{a}{2}g_l{}^3 + C_l{}^4 A_0{}^\circ + C_l{}^5 A_0{}^2$$

$$\Phi_l = \frac{1}{r_3}\left(\lambda_{11}\Phi_l{}^3 - \Phi_l{}^1 + \frac{2\lambda_4'}{\pi}\sum_{n=0}^{\infty} B_{n,\,l}\Phi_n{}^1\right)$$

$$\lambda = \frac{k_2\lambda_2'}{k_2 - \lambda_{21}}, \quad r_1 = \frac{\lambda_4' - \lambda_1' - \lambda_3'\lambda_{11}}{r_3}, \quad r_2 = \frac{2\lambda_1'\lambda_4'}{\pi r_3}, \quad r_3 = \lambda_{31}\lambda_{11} - 1$$

$$C_l{}^1 = C_l - \lambda_1' B_{-1,\,l}, \quad C_l{}^2 = -\lambda_{11}C_l, \quad C_l{}^3 = -C_l - \lambda_2' B_{-1,\,l}$$

$$C_l{}^4 = \lambda_3' B_{-1,\,l} - \lambda_{31}C_l$$

$$C_l{}^5 = \lambda_4' B_{-1,\,l} - C_l, \quad C_l = \pi\sin\frac{l\pi}{2}\left(\operatorname{tg}\frac{a}{4}\right)^{l+1}, \quad \lambda_i' = 2\operatorname{tg}\frac{a}{2}\lambda_i$$

$$A_0{}^i = \frac{1}{\pi}\cos\frac{a}{2}A^i$$

$$g_l{}^i = \int_{-1}^{1} G_i\left[2\operatorname{arctg}\left(x\operatorname{tg}\frac{a}{2}\right)\right]U_l(x)\sqrt{1-x^2}\left(1 + x^2\operatorname{tg}^2\frac{a}{2}\right)^{-1}dx$$

$$G_1(x) = F_1(x) + \frac{\lambda_1'}{2a}A^2, \quad G_2(x) = F_2(x) + \frac{\lambda_2'}{2a}A^3$$

$$G_3(x) = F_3(x) - \frac{1}{2a}(\lambda_3' A^2 + \lambda_4' A^4)$$

$$B_{m-1,\,l} = -8(l+1)\operatorname{ctg}\frac{a}{2}\sum_{j=0}^{\infty}(-1)^j(2j+1)\left(\operatorname{tg}\frac{a}{4}\right)^{2j+1}\times$$

$$\left[\sin\frac{(m+l)\pi}{2}\right]^2[m^2 - (2j-l)^2]^{-1}[m^2 - (2j+l+2)^2]^{-1}$$

$$i = 1, 2, 3, 4; \; m, l = 0, 1, 2, \ldots$$

($T_n(x)$, $U_n(x)$ are Chebyshev polynomials of the first and second kind).

Using the estimates presented in [11], it can be shown that the systems (3.3), (3.4) are at least quasi-regular for any geometric and physical parameters of the problem and the method of reduction is completely applicable for their solution.

The stress intensity coefficients at the endfaces of the inclusions are determined by the formulas

$$\{k_{-1i}^{\pm},\ k_{-2i}^{\pm},\ k_{-3i}^{\pm}\} = \lim_{x\to\pm a\pm 0}\left[\sqrt{\frac{x}{a}-1}\,\{\sigma_{yi}(x),\ \tau_{xyi}(x),\ \sigma_{xi}(x)\}\right] \qquad (3.6)$$

$$\{k_{+1i}^{\pm},\ k_{+2i}^{\pm},\ k_{+3i}^{\pm}\} = \lim_{x\to\pm a\mp 0}\left[\sqrt{1-\frac{x}{a}}\,\{\sigma_{yi}(x),\ \tau_{xyi}(x),\ \sigma_{xi}(x)\}\right]$$

Using the formula

$$\frac{1}{2\pi}\int_{-a}^{a}\operatorname{ctg}\left(\frac{t-x}{2}\right)T_k\left(\operatorname{tg}\frac{t}{2}\operatorname{ctg}\frac{a}{2}\right)\sec\frac{t}{2}\,[2(\cos t-\cos a)]^{-1}dt =$$

$$J_{1,k}(x) - \operatorname{sign}(x)\,T_k\left(\operatorname{tg}\frac{x}{2}\operatorname{ctg}\frac{a}{2}\right)\sec\frac{x}{2}\,[2(\cos a-\cos x)]^{-1}$$

$$a<|x|\leqslant\pi,\quad k=0,1,\ldots$$

$$J_{1,0}(x)=\frac{1}{2}\sec\frac{a}{2}\operatorname{tg}\frac{x}{2},\quad J_{1,k}(x)=\frac{1}{2}\operatorname{cosec}\frac{a}{2}\left(\sec\frac{x}{2}\right)^2U_{k-1}\times$$

$$\left(\operatorname{tg}\frac{x}{2}\operatorname{ctg}\frac{a}{2}\right),\quad k=1,2,\ldots$$

we obtain from the relationships (3.6)

$$k_{-1i}^{\pm}=\mp m_{12}^{-}K_2^{\pm}\pm l_1^{+}K_4^{\pm},\quad k_{-2i}^{\pm}=\pm m_{12}^{-}K_1^{\pm}\pm l_1^{\pm}K_3^{\pm} \qquad (3.7)$$

$$k_{-31}^{\pm}=\mp n_{12}^{+}K_2^{\pm}\pm r_{12}^{-}K_4^{\pm},\quad k_{-32}^{\pm}=\mp n_{21}^{+}K_2^{\pm}\pm r_{21}^{-}K_4^{\pm}$$

$$k_{+11}^{\pm}=m_{12}^{+}K_1^{\pm},\quad k_{+12}^{\pm}=-m_{21}^{+}K_1^{\pm},\quad k_{+21}^{\pm}=m_{12}^{+}K_2^{\pm},\quad k_{+22}^{\pm}=-m_{21}^{+}K_2^{\pm}$$

$$k_{+31}^{\pm}=n_{12}^{-}K_1^{\pm}+r_{12}^{+}K_3^{\pm},\quad k_{+32}^{\pm}=-n_{21}^{-}K_1^{\pm}-r_{21}^{+}K_3^{\pm}$$

$$K_j^{\pm}=\sec\frac{a}{2}(2a\sin a)^{-1}\sum_{n=0}^{\infty}(\pm 1)^nA_n^j,\quad i=1,2;\ j=1,2,3,4$$

In the case of symmetry of the external load relative to the coordinate axes

$$\Phi_l^j=2\sin\frac{a}{2}g_l^j,\quad G_j(x)=F_j(x),\quad F_2(x)\equiv 0,\quad j=1,2,3,4;\quad l=0,1,\ldots$$

the functions $F_1(x)$, $F_3(x)$ are even and the systems (3.3), (3.4) as well as the relations (3.5) yield $A_k^1=A_k^3=A_{2k}^2=A_{2k}^4=0,\quad k=0,1,\ldots$

Hence, we obtain $f_1(x)=f_3(x)=0,\ |x|\leqslant a$ in particular.

Denoting the stress intensity coefficients for the case of force and geometrical symmetry of the problem as

$$k_1=k_{-1i}^{\pm},\ k_2=\pm k_{+21}^{\pm}=\mp k_{+22}^{\pm},\ k_3=k_{-3i}^{\pm}\quad (k_{+1i}^{\pm}=k_{-2i}^{\pm}=k_{+3i}^{\pm}=0)$$

we find from (3.7)

$$k_1=-m_{12}^{-}K_2+l_1^{+}K_4,\quad k_2=m_{12}^{+}K_2,\quad k_3=-n_{12}^{+}K_2+r_{12}^{-}K_4$$

$$K_j=\sec\frac{a}{2}(2a\sin a)^{-1}\sum_{n=0}^{\infty}A_{2n+1}^j,\quad j=2,4$$

The coefficients A_{2n+1}^j $(j=2,4)$ are found from (3.3) and (3.5) for $p=0$.

A numerical analysis of the problem was carried out on the M-222 electronic computer for two symmetric loading cases:

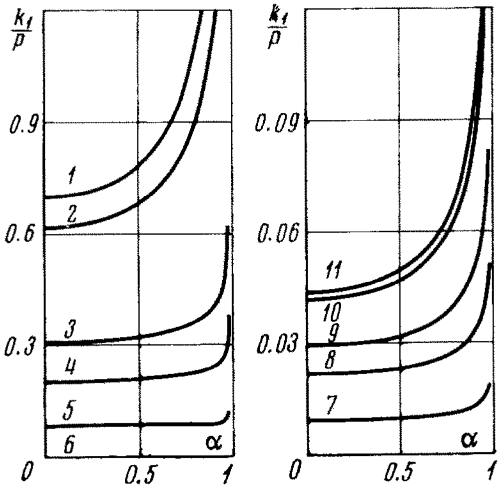

Fig. 2

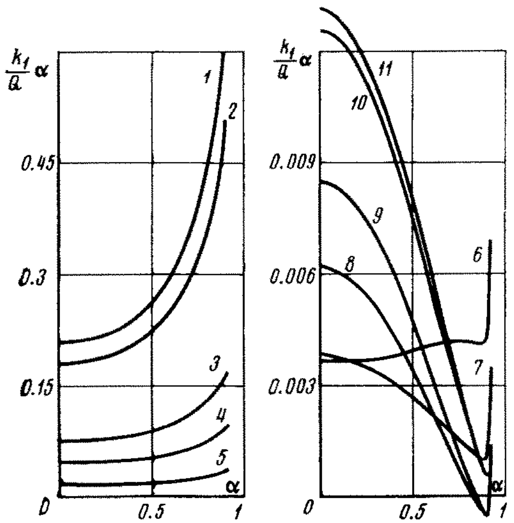

Fig. 3

1) $\sigma_y^\infty = P$, $\sigma_{xi}^\infty = P_i = 0$ $(i = 1, 2)$, no concentrated forces and moments;

2) loading is by the concentrated forces $\pm iQ$, applied at the points $\pm ia - 2n\pi$, respectively $(n = \ldots -2, -1, 0, 1, 2, \ldots)$.

The system (3.3) is solved (for $p = 0$) by the method of reduction, i.e. a finite number of terms N was taken into account in the expansion (3.2), and consequently a corresponding truncated system of linear algebraic equations was considered. The convergence was checked by two methods: by comparing the functions $f_j^N(x)$ $(j = 2, 4)$ evaluated for $N = M$ and $N = 2M$, respectively, and by comparing the accuracy of satisfying the initial system of integral equations (2.6) in the case (3.1) by the functions $f_j^N(x)$ found. In the examples considered, depending on the relative length of the inclusion $\alpha = a/\pi$, it turned out to be possible to limit oneself to the solution of a system of equations from the 15-th to the 50-th order to achieve a 1% accuracy in the calculations.

The calculations were performed for $a/h = 10$, $\nu_0 = \nu = 1/3$.

Figures 2 and 3 illustrate the dependence of the normal stress intensity coefficient k_1 on the relative length of the inclusion α in cases 1 and 2, resepctively. The curves 1 – 11 correspond to the following values of the relative stiffness of the inclusions $k = E_0/E$: 0.001, 0.01, 0.1, 0.2, 0.5, 1, 2, 5, 10, 100, 1 000. The results represented by curves 1 – 11 differ from the solutions of periodic problems for slits and absolutely stiff inclusions, respectively, by not more than 1 – 2%. When α is almost zero, we arrive at the results for one isolated inclusion.

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FREQUENCY SPECTRA AND MODES OF FREE VIBRATIONS OF DOUBLY PERIODIC SYSTEMS

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An analog of the finite element method is proposed for the solution of natural vibrations problems for doubly-periodic systems. The approximate solution is constructed for each separate element. The infuence of adjacent elements is taken into account by the introduction of force factors and matching conditions. Numerical examples are analyzed.

1. Let the doubly-periodic system be generally referred to an oblique $0x_1x_2$ coordinate system so that the properties of the system are repeated for a displacement a_1 along the $0x_1$ axis and a_2 along $0x_2$. Let us consider the vibrations of a single element bounded by the lines $x_1' = 0$, $x_1' = a_1$, $x_2' = 0$, $x_2' = a_2$ in a local coordinate system. Let us represent the dicplacement vector for the vibrations mode as a series expansion in a system of coordinate functions

$$\mathbf{u}^{(kl)} = \sum_{n=1}^{N} C_n^{(kl)} \mathbf{v}^{(n)} \tag{1.1}$$